1.5.5 Lec 35 (II): Scalar Wave Equations (Acoustics)

In this section we discuss the general solution to the wave equation. The wave equation has two forms: scalar waves (acoustics) and vector waves (electromagnetics). These have an important mathematical distinction but a similar solution space, one scalar and the other vector. To understand the differences we start with the scalar wave equation.

The scalar wave equation: A good starting point for understanding PDEs is to explore the scalar wave equation (Eq. 1.28, p. 73). Thus, we shall limit our analysis to acoustics, the classic case of scalar waves. Acoustic wave propagation was first analyzed mathematically by Isaac Newton (electricity had yet to be discovered) in his famous book *Principia* (1687), in which he first calculated the speed of sound based on the conservation of mass and momentum.

Early history: The study of wave propagation begins at least as early as Huygens (ca. 1678), followed soon after (ca. 1687) by Sir Isaac Newton's calculation of the speed of sound (Pierce, 1981, p. 15). The acoustic variables are the *pressure*

$$p(\boldsymbol{x}, t) \leftrightarrow P(\boldsymbol{x}, \omega)$$

and the *particle velocity*

$$u(\boldsymbol{x}, t) \leftrightarrow U(\boldsymbol{x}, \omega).$$

To obtain a wave, one must include two basic components: the stiffness of air, and its mass. These two equations shall be denoted (1) *Newton's* 2^{nd} *law* ($F = ma$) and (2) *Hooke's law* ($F = kx$), respectively. In vector form these equations are (1) *Euler's equation* (i.e., conservation of momentum density)

$$-\nabla p(\boldsymbol{x}, t) = \rho_o \frac{\partial}{\partial t} \boldsymbol{u}(\boldsymbol{x}, t) \leftrightarrow \rho_o s U(\boldsymbol{x}, s), \tag{1.114}$$

which assumes the time-average density ρ_o to be independent of time and position $\boldsymbol{x}$, and (2) the *continuity equation* (i.e., conservation of mass density)

$$-\nabla \cdot \boldsymbol{u}(\boldsymbol{x}, t) = \frac{1}{\eta_o P_o} \frac{\partial}{\partial t} p(\boldsymbol{x}, t) \leftrightarrow \frac{s}{\eta_o P_o} P(\boldsymbol{x}, s) \tag{1.115}$$

(Pierce, 1981; Morse, 1948, p. 295). Here $P_o = 10^5$ [Pa] is the barometric pressure, $\eta_o P_o$ is the dynamic (adiabatic) stiffness, with $\eta_o = 1.4$. Combining Eqs. 1.114 and 1.115 (removing $\boldsymbol{u}(\boldsymbol{x}, t)$) results in the 3-dimensional (3D) scalar pressure wave equation

$$\nabla^2 p(\boldsymbol{x}, t) = \frac{1}{c_o^2} \frac{\partial^2}{\partial t^2} p(\boldsymbol{x}, t) \leftrightarrow \frac{s^2}{c_o^2} P(\boldsymbol{x}, s) \tag{1.116}$$

with $c_o = \sqrt{\eta_o P_o / \rho_o}$ being the sound velocity. Because the merged equations describe the pressure, which is a scalar field, this is an example of the *scalar wave equation*.

Exercise: Show that Eqs. 1.114 and 1.115 can be reduced to Eq. 1.116. **Solution:** Taking the divergence of Eq. 1.114 gives

$$-\nabla \cdot \nabla p(\boldsymbol{x}, t) = \rho_o \frac{\partial}{\partial t} \nabla \cdot \boldsymbol{u}(\boldsymbol{x}, t). \tag{1.117}$$

Note that $\nabla \cdot \nabla = \nabla^2$ (p. 391). Next, substituting Eq. 1.115 into the above relation results in the scalar wave equation Eq. 1.116, since $c_o = \sqrt{\eta_o P_o / \rho_o}$.

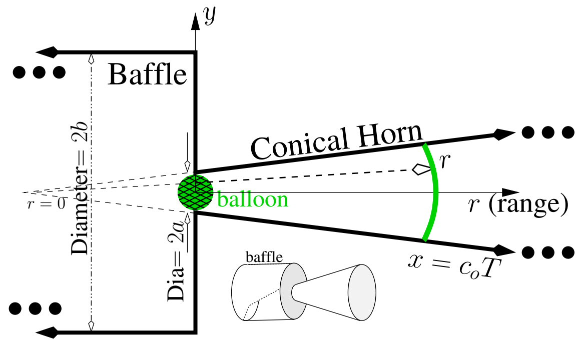

Figure 1.19: *Experimental setup showing a large pipe on the left terminating the wall containing a small hole with a balloon, shown in green. At time $t = 0$ the balloon is pricked and a pressure pulse is released. The baffle on the left represents a semi-∞ long tube having a large radius compared to the horn input diameter $2a$, such that the acoustic admittance looking to the left ($A/\rho_o c_o$ with $A \to \infty$) is very large compared to the horn's throat admittance (Eq. 1.139). At time T the outbound pressure pulse $p(r, T) = \delta(t - x/c_o)/r$ has reached a radius $x = r - r_o = c_o T$ where $r = x$ is the location of the source at the throat of the horn and r is measured from the vertex. At the throat of the horn $\mathcal{V}_+/\boldsymbol{A}_+ = \mathcal{V}_-/\boldsymbol{A}_-$.*

1.5.6 Lec 36a: The Webster horn equation (I)

There is an important generalization of the problem of lossless plane-wave propagation in 1-dimensional (1D) uniform tubes known as *transmission line theory*. By allowing the area $A(r)$ of the horn to vary along the *range* axis r (the direction of wave propagation), as depicted in Fig. 1.19 for the *conical horn* (i.e., $A(r) = A_o(r/r_o)^2$), general solutions to the wave equation may be explored. Classic applications of horns include vocal tract acoustics, loudspeaker design, cochlear mechanics, the hydrogen atom, and cases having wave propagation in periodic media (Brillouin, 1953).

For the 1D scalar wave equation (guided waves, aka, acoustic horns), the *Webster Laplacian* is

$$\nabla_r^2\, \varrho(r,t) = \frac{1}{A(r)}\frac{\partial}{\partial r}\left[A(r)\frac{\partial}{\partial r}\right]\varrho(r,t). \tag{1.118}$$

The Webster Laplacian is based on the *quasi-static approximation* (P10: p. 141) which requires that the frequency lie below the critical value $f_c = c_o/2d$, namely that a half wavelength be greater than the horn diameter d (i.e., $d < \lambda/2$).[123] For the case of the adult human ear canal, $d = 7.5$ [mm], $f_c = (343/2 \cdot 7.5) \times 10^{-3} \approx 22.87$ [kHz].

The term on the right of Eq. 1.118, which is identical to Eq. 1.101 (p. 194), is also the Laplacian for thin tubes (e.g., rectangular, spherical, and cylindrical coordinates). Thus the Webster horn "wave" equation is

$$\frac{1}{A(r)}\frac{\partial}{\partial r}\left[A(r)\frac{\partial}{\partial r}\right]\varrho(r,t) = \frac{1}{c_o^2}\frac{\partial^2}{\partial t^2}\varrho(r,t) \leftrightarrow \frac{s^2}{c_o^2}\mathcal{P}(r,s), \tag{1.119}$$

where $\varrho(r,t) \leftrightarrow \mathcal{P}(r,s)$ is the *acoustic pressure* in Pascals [Pa] (Hanna and Slepian, 1924; Mawardi, 1949; Morse, 1948); Olson (1947, p. 101); Pierce (1981, p. 360). Extensive experimental analyses for various types of horns (conical, exponential, parabolic) along with a review of horn theory may be found in Goldsmith and Minton (1924).

The limits of the Webster horn equation: It is commonly stated that the Webster horn equation (WHEN) is fundamentally limited and thus is an approximation that only applies to frequencies much lower than f_c. However, in all these discussions it is assumed that the area

[123] This condition may be written several ways, the most common being $ka < 1$, where $k = 2\pi/\lambda$ and a is the horn radius. This may be expressed in terms of the diameter as $\frac{2\pi}{\lambda}\frac{d}{2} < 1$, or $d < \lambda/\pi < \lambda/2$. Thus $d < \lambda/2$ may be a more precise metric by the factor $\pi/2 \approx 1.6$. This is called the *half-wavelength assumption*, a synonym for the quasi-static approximation.

function $A(r)$ is the horn's cross-sectional area, not the area of the iso-pressure wave-front (Morse, 1948; Shaw, 1970; Pierce, 1981).

In the next section it is shown that this "limitation" may be avoided (subject to the $f < f_c$ quasi-static limit (P10, p. 142)), *making the Webster horn theory an "exact" solution for the lowest-order "plane-wave" eigenfunction*. The nature of the quasi-static approximation is that it "ignores" higher-order evanescent modes, which are naturally small since they are in cutoff (evanescent modes do not propagate) (Hahn, 1941; Karal, 1953). This is the same approximation that is required to define an impedance, since every eigenmode defines an impedance (Miles, 1948).

To apply this theory, the acoustic variables (eigenfunctions) are redefined for the *average pressure* and its corresponding *volume velocity*, each defined on the iso-pressure wave-front boundary (Webster, 1919; Hanna and Slepian, 1924). The resulting acoustic impedance is then the ratio of the average pressure over the volume velocity. This approximation is valid up to the frequency where the first cross mode begins to propagate ($f > f_c$), which may be estimated from the roots of the Bessel eigenfunctions (Morse, 1948). Perhaps it should be noted that these ideas, which come from acoustics, apply equally well to electromagnetics, or any other wave phenomena described by eigenfunctions.

The best known examples of wave propagation are electrical and acoustic *transmission lines*. Such systems are loosely referred to as the *telegraph* or *telephone equations*, referring back to the early days of their discovery (Heaviside, 1892; Campbell, 1903b; Brillouin, 1953; Feynman, 1970a). In acoustics, waveguides are known as horns, such as the horn connected to the first phonographs from around the turn of the century (Webster, 1919). Thus the names reflect the historical development, to a time when the mathematics and the applications were running in close parallel.

1.5.7 Lec 36b: Webster horn equation (II): Derivation

In this section we transform the acoustic equations Eq. 1.114 and 1.115 (p. 211) into their equivalent integral form Eq. 1.119 (p. 212). This derivation is similar (but not identical) to that of Hanna and Slepian (1924) and Pierce (1981, p. 360).

Conservation of momentum: The first step is an integration of the normal component of Eq. 1.114 (p. 211) over the iso-pressure surface $\mathcal{S}$, defined by $\nabla p = 0$

$$-\int_{\mathcal{S}} \nabla p(\mathbf{x}, t) \cdot d\mathbf{A} = \rho_o \frac{\partial}{\partial t} \int_{\mathcal{S}} \mathbf{u}(\mathbf{x}, t) \cdot d\mathbf{A}.$$

The *average pressure* $\varrho(x,t)$ is defined by dividing by the total area

$$\varrho(x,t) \equiv \frac{1}{A(x)} \int_{\mathcal{S}} p(x,t)\, \hat{\boldsymbol{n}} \cdot d\boldsymbol{A}. \tag{1.120}$$

From the definition of the gradient operator

Is there an isopressure surf
diffusion?

$$\nabla p = \frac{\partial p}{\partial x} \hat{\boldsymbol{n}}, \tag{1.121}$$

where $\hat{\boldsymbol{n}}$ is a unit vector perpendicular to the iso-pressure surface $\mathcal{S}$. Thus the left side of Eq. 1.114 reduces to $\partial \varrho(x,t)/\partial x$.

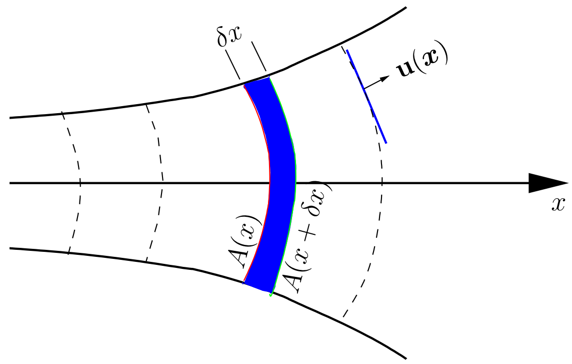

Figure 1.20: *Derivation of horn equation using Gauss's law: The divergence of the velocity $\nabla \cdot \mathbf{u}$, within δx, shown as the filled shaded region, is integrated over the enclosed volume. Next the divergence theorem is applied, transforming the integral to a surface integral normal to the surface of propagation. This results in the difference of the two volume velocities $\delta\nu = \nu(x+\delta x) - \nu(x) = [\mathbf{u}(x+\delta x)\cdot \mathbf{A}(x+\delta x) - \mathbf{u}(x)\cdot \mathbf{A}(x)]$. The flow is always perpendicular to the iso-pressure contours.*

The integral on the right side defines the *volume velocity*

$$\nu(x,t) \equiv \int_S \mathbf{u}(x,t)\cdot d\boldsymbol{A}. \tag{1.122}$$

Thus the integral form of Eq. 1.114 becomes

$$-\frac{\partial}{\partial x}\varrho(x,t) = \frac{\rho_o}{A(x)}\frac{\partial}{\partial t}\nu(x,t) \leftrightarrow \mathscr{Z}(x,s)\,\mathcal{V}, \tag{1.123}$$

where

$$\mathscr{Z}(s,x) = s\rho_o/A(x) = sM(x) \tag{1.124}$$

and $M(x) = \rho_o/A(x)$ [kgm/m^5] is the per-unit-length mass density of air.

Conservation of mass: Integrating Eq. 1.115 (p. 211) over the volume V gives

$$-\int_V \nabla\cdot\boldsymbol{u}\,dV = \frac{1}{\eta_o P_o}\frac{\partial}{\partial t}\int_V p(\boldsymbol{x},t)dV.$$

The volume V is defined by two iso-pressure surfaces between x and $x+\delta x$ (green region of Fig. 1.20). On the right-hand side we use the definition of the *average pressure* (i.e., Eq. 1.120), integrated over the volume dV.

Applying Gauss's law to the left-hand side,[124] and using the definition of ϱ (on the right), in the limit $\delta x \to 0$, gives

$$\frac{\partial\nu}{\partial x} = -\frac{A(x)}{\eta_o P_o}\frac{\partial\varrho}{\partial t} \leftrightarrow -\mathscr{Y}(x,s)\,\mathcal{P}(x,s) \tag{1.125}$$

where

$$\mathscr{Y}(s,x) = sA(x)/\eta_o P_o = sC(x).$$

$C(x) = A(x)/\eta_o P_o$ [m^4/N] is the per-unit-length compliance of the air. Equations 1.123 and 1.125 accurately characterize the Webster plane-wave mode up to the frequency where the higher-order eigenmodes begin to propagate (i.e, $f > f_c$).

Speed of sound c_o: In terms of $M(x)$ and $C(x)$, the speed of sound and the acoustic admittance are

$$c_o = \sqrt{\frac{\textit{stiffness}}{\textit{mass}}} = \frac{1}{\sqrt{C(x)M(x)}} = \sqrt{\frac{\eta_o P_o}{\rho_o}}. \tag{1.126}$$

[124] As shown in Fig. 1.20, we convert the divergence into the difference between two volume velocities, namely $\nu(x+\delta x) - \nu(x)$, and $\partial\nu/\partial x$ as the limit of this difference over δx, as $\delta x \to 0$.

Characteristic admittance $\mathcal{Y}(x)$: Since the horn equation (Eq. 1.119) is second-order, it has pressure eigenfunction solutions $\mathcal{P}^+$ and $\mathcal{P}^-$ and their corresponding velocity eigenfunctions $\mathcal{V}^+$ and $\mathcal{V}^-$, related through Eq. 1.123, which defines the *characteristic admittance* $\mathcal{Y}(x)$

$$\mathcal{Y}(x) = \frac{1}{\sqrt{\textit{stiffness} \cdot \textit{mass}}} = \sqrt{\frac{C(x)}{M(x)}} = \frac{A(x)}{\rho_o c_o} = \frac{\mathcal{V}^+}{\mathcal{P}^+} = \frac{\mathcal{V}^-}{\mathcal{P}^-} \tag{1.127}$$

(Campbell, 1903a, 1910, 1922). The *characteristic impedance* $\mathcal{Z}(x) = 1/\mathcal{Y}(x)$. Based on physical requirements, the admittance must be positive, since only the positive square root is allowed. Note that once the pressure eigenfunctions are determined, the corresponding velocity is given by Eq. 1.123, and then the admittance is given by the ratio of the two. It follows that the characteristic admittance is given by the logarithmic derivative of the pressure eigenfunctions.

Since the horn (Eq. 1.119) is lossless, $\mathcal{Y}(x)$ must be real and positive. If losses are introduced, the *propagation function* (p. 152)

$$\kappa(s) = \pm \frac{s}{c_o} \tag{1.128}$$

and the characteristic impedance $\mathcal{Y}(x, s)$ will become complex analytic functions of the Laplace frequency s (Kirchhoff, 1974; Mason, 1928; Ramo *et al.*, 1965; Pierce, 1981, p. 532-4).

One must be careful in defining the area $A(x)$: The area is *not* the cross-sectional area of the horn, rather it is the wave-front area, as discussed next. Since $A(x)$ is independent of frequency, it is independent of the wave direction.

1.5.8 Matrix formulation of WHEN (III)

Newton's laws of conservation of momentum (Eq. 1.114) and mass (Eq. 1.115) are modern versions of Newton's starting point for calculating the horn lowest-order plane-wave eigenmode wave speed. Following the derivation of the acoustic equations for the average pressure $\mathcal{P}(r, \omega)$ and the volume velocity (Eqs. 1.123, 1.125), the equations may be expressed as the 2x2 matrix

$$-\frac{d}{dr}\begin{bmatrix}\mathcal{P}(r,\omega)\\ \mathcal{V}(r,\omega)\end{bmatrix} = \begin{bmatrix}0 & sM(r)\\ sC(r) & 0\end{bmatrix}\begin{bmatrix}\mathcal{P}(r,\omega)\\ \mathcal{V}(r,\omega)\end{bmatrix}. \tag{1.129}$$

Here $M(r) = \rho_o/A(r)$ and $C(r) = A(r)/\eta_o P_o$ are the unit-length mass and compliance of the horn (Ramo *et al.*, 1965, p. 213). The product of $\mathcal{P}(r, \omega)$ and $\mathcal{V}(r, \omega)$ define the acoustic power while their ratio defines an impedance (Pierce, 1981, p. 37-41).

To obtain the Webster horn pressure equation Eq. 1.119 (p. 212) from Eq. 1.129, take the partial derivative of the top equation

$$-\frac{\partial^2 \mathcal{P}}{\partial r^2} = s\frac{\partial M(r)}{\partial r}\mathcal{V} + sM(r)\frac{\partial \mathcal{V}}{\partial r},$$

and use the lower equation to remove $\partial \mathcal{V}/\partial r$

$$\frac{\partial^2 \mathcal{P}}{\partial r^2} + s\frac{\partial M(r)}{\partial r}\mathcal{V} = s^2 M(r)C(r)\mathcal{P} = \frac{s^2}{c_o^2}\mathcal{P}.$$

Then use the upper equation a second time to remove $\mathcal{V}$

$$\frac{\partial^2}{\partial r^2}\mathcal{P} + \frac{1}{A(r)}\frac{\partial A(r)}{\partial r}\frac{\partial}{\partial r}\mathcal{P} = \frac{s^2}{c_o^2}\mathcal{P}(r, s). \tag{1.130}$$

By use of the chain rule, equations of this form may be directly integrated

$$\begin{aligned}\nabla_r \mathcal{P} &= \frac{1}{A(r)}\frac{\partial}{\partial r}\left[A(r)\frac{\partial}{\partial r}\right]\mathcal{P}(r,s)\\ &= \frac{\partial^2}{\partial r^2}\mathcal{P}(r,s) + \frac{1}{A(r)}\frac{\partial A(r)}{\partial r}\,\mathcal{P}_r(r,s). \end{aligned} \tag{1.131}$$

This is equivalent to *integration by parts*, with integration factor $A(r)$. Above we may replace s^2/c_o^2 everywhere by $\kappa(s)$.

Merging Eqs. 1.130 and 1.131 results in the Webster horn equation (Eq. 1.119, p. 212):

$$\frac{1}{A(r)}\frac{\partial}{\partial r}A(r)\frac{\partial}{\partial r}\mathcal{P}(r,s) = \frac{s^2}{c_o^2}\mathcal{P}(r,s).$$

Equations having this integrated form are known as *Sturm-Liouville equations*. This important class of differential equations follows from the use of separation of variables on the Laplacian, in any (i.e., every) separable coordinate system (Morse and Feshbach, 1953, Ch. 5.1, p. 494-523).

Summary: Applying Gauss's law to the 3D wave equation (Eq. 1.116, p. 211) results in a 1D Webster horn equation (WHEN, Eq. 1.119, p. 212), which is a non-singular Sturm-Liouville equation, where the area function is the integration factor $A(r)$.[125]

Thus Eqs. 1.116 and 1.129 are equivalent to the WHEN (Eq. 1.119).

1.5.9 Lec 37a: d'Alembert's eigenvector superposition principle

Since the Webster horn equation is second-order in time, it has two unique pressure eigenfunctions $\mathcal{P}^+(r,s)$ and $\mathcal{P}^-(r,s)$. The general solution may always be written as the superposition of pressure eigenfunctions, with amplitudes determined by the boundary conditions.

Based on *d'Alembert's superposition principle*, the pressure $\mathcal{P}$ and volume velocity $\mathcal{V}$ may be decomposed in terms of the pressure eigenfunctions $\mathcal{P}^+$ and $\mathcal{P}^-$

$$\begin{bmatrix}\mathcal{P}(r,\omega)\\ \mathcal{V}(r,\omega)\end{bmatrix} = \begin{bmatrix}1 & 1\\ \mathcal{Y}(r) & -\mathcal{Y}(r)\end{bmatrix}\begin{bmatrix}\mathcal{P}^+(r,\omega)\\ \mathcal{P}^-(r,\omega)\end{bmatrix} \tag{1.132}$$

where $\mathcal{Y} = A(r)/\rho_o c_o = 1/\mathcal{Z}$. This equation has several important applications.

Generalized admittance/impedance: The *generalized admittance*[126] $Y_{in}(r,s)$ looking into the horn is

$$Y_{in}(r,s) \equiv \frac{\mathcal{V}(r,\omega)}{\mathcal{P}(r,\omega)} = \frac{\mathcal{V}^+ - \mathcal{V}^-}{\mathcal{P}^+ + \mathcal{P}^-} \tag{1.133}$$

$$= \frac{\mathcal{V}_+}{\mathcal{P}_+}\left(\frac{1-\mathcal{V}^-/\mathcal{V}^+}{1+\mathcal{P}^-/\mathcal{P}^+}\right) \tag{1.134}$$

$$= \mathcal{Y}(r)\frac{1-\Gamma(r,s)}{1+\Gamma(r,s)}. \tag{1.135}$$

[125] The Webster horn equation is also related to Schrödenger's equation, the cornerstone of quantum mechanics (Sect. 1.5.6, p. 449).

[126] It is "generalized" in the sense that it is not a Brune, rational function, impedance.

Table 1.10: *Table of horns and their properties for $N = 1$, 2 or 3 dimensions, along with the exponential horn (*EXP*). In this table the horn's range variable is ξ [m], having* area $A(\xi)$ [m^2]*, diameter $\xi_o = \sqrt{A(\xi_o)/4\pi}$ [m]. $F(r)$ is the coefficient on $\mathcal{P}_x$, $\kappa(s) \equiv s/c_o$, where c_o is the speed of sound and $s = \sigma + \omega\jmath$ is the Laplace frequency. The range variable ξ may be rendered dimensionless (see Fig. 1.21) by normalizing it as $r \equiv (\xi - \xi_o)/(L - \xi_o)$, with ξ the linear distance along the horn axis, from ξ_o to L corresponding to $r = 0$ to 1. The horn's eigenfunctions are $\mathcal{P}^\pm(\xi, \omega) \leftrightarrow \varrho^\pm(\xi, t)$. When $\pm$ is indicated, the outbound solution corresponds to the negative sign. Eigenfunctions $H_o^\pm(\xi, s)$ are outbound and inbound Hankel functions. The last column is the* radiation admittance *normalized by the* characteristic admittance $\mathcal{Y}(r) = A(r)/\rho_o c_o$.

N	Name	radius	Area/A_o	$F(r)$	$\mathcal{P}^\pm(r,s)$	$\varrho^\pm(r,t)$	$Y^\pm_{rad}/\mathcal{Y}$
1D	uniform	1	1	0	$e^{\pm\kappa(s)r}$	$\delta(t \mp r/c_o)$	1
2D	parabolic	$\sqrt{r/r_o}$	r/r_o	$1/r$	$H_o^\pm(-\jmath\kappa(s)r)$	—	$\frac{-\jmath r H_1^\pm}{H_o^\pm}$
3D	conical	r	r^2	$2/r$	$e^{\pm\kappa(s)r}/r$	$\delta(t) \pm \frac{c_o}{r}u(t)$	$1 \pm c_o/sr$
EXP	exponential	e^{mr}	e^{2mr}	$2m$	$e^{-\left(m\pm\sqrt{m^2+\kappa^2}\right)r}$	$e^{-mr}E(t)$	Eq. 1.155

Here we have factored out the forward traveling eigenfunction $\mathcal{V}^+$ and $\mathcal{P}^+$, and re-expressed Y_{in} in terms of two ratios, the *characteristic admittance* $\mathcal{Y}(r)$ (Eq. 1.127) and the reflectance $\Gamma(r,s)$. $Y_{in}(s)$ depends on the entire horn, and in the case of a finite length horn it depends on the terminating admittance. When the horn is terminated, reflections occur, resulting in the horn having poles and zeros at frequencies $s_k \in \mathbb{C}$, where $\Gamma(r, s_k) = \pm 1$.

The *reflectance* is defined as

$$\Gamma(r,s) \equiv \frac{\mathcal{V}^-(r,\omega)}{\mathcal{V}^+(r,\omega)} = \frac{\mathcal{P}^-(r,\omega)}{\mathcal{P}^+(r,\omega)} \tag{1.136}$$

$$= \frac{Z_{in}(r,s) - \mathcal{Z}}{Z_{in}(r,s) + \mathcal{Z}}, \tag{1.137}$$

which follows by a rearrangement of terms in Eq. 1.127. The magnitude of the reflections depends on $|\Gamma|$, which must be between 0 and 1. Alternatively, this equation may be obtained by solving Eq. 1.135 for $\Gamma(r,s)$.

Horn radiation admittance: The horn's acoustic input admittance $Y_{in}(r,s)$ is defined by dividing Eq. 1.123 (p. 214) by $\mathcal{P}^\pm$

$$Y^\pm_{in}(r,s) = \frac{\mathcal{V}(r,s)}{\mathcal{P}(r,s)} = -\frac{A(r)}{s\rho_o}\frac{d}{dr}\ln \mathcal{P}^\pm(r,s). \tag{1.138}$$

A horn's acoustic *radiation admittance* $Y^\pm_{rad}(r,s)$ is the input admittance (Eq. 1.135) when there is no terminating load[127]

$$Y^\pm_{rad}(r,s) = \lim_{r\to\infty} Y^\pm_{in}(r,s). \tag{1.139}$$

The input admittance becomes the radiation admittance when the horn is infinite in length, namely it is the input admittance for an eigenfunction. A table of properties is given in Table 1.10 for four different simple horns. Expressions for $Y_{rad}(x,s)$ are given in the last column of Table 1.10.

[127] To compute the radiation impedance $Y^\pm_{rad}$ one must know the eigenfunctions $\mathcal{P}^\pm(r,s)$.

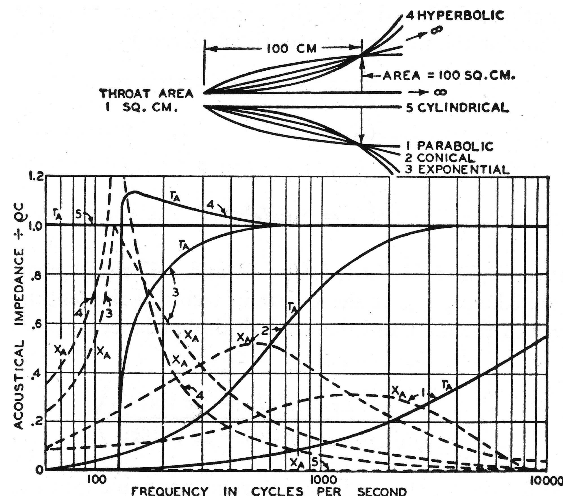

Figure 1.21: *Throat acoustical resistance r_A and acoustical reactance x_A, frequency characteristics of infinite eigenfunctions of the parabolic, conical, exponential, hyperbolic and cylindrical horns, having a throat area of 1 [cm^2]. Note how the "critical" frequency (defined here as the frequency where the reactive and real parts of the radiation impedance are equal) of the horn reduces dramatically with the type of horn. For the uniform horn, the reactive component is zero, so there is no cutoff frequency. For the parabolic horn (1) the cutoff is around 3 kHz. For the conical horn (2) the cutoff is at 0.6 [kHz]. For the exponential horn (3) the critical frequency is around 0.18 [kHz], which is one 16th that of the parabolic horn. For each horn the cross-sectional area is defined as 100 [cm^2] at a distance of 1 [m] from the throat (Olson, 1947, p. 101), (Morse, 1948, p. 283).*

1.5.10 Lec37b: Complex analytic $\Gamma(s)$ and $Y_{in}(s)$

When defining the complex reflectance $\Gamma(s)$ as a function of the complex frequency $s = \sigma + j\omega$, a *very* important assumption has been made: even though $\Gamma(s)$ is defined by the ratio of two functions of real (radian) frequency ω, like the impedance, the reflectance must be *causal* (postulate P1, p. 140). Namely $\Gamma(s) \leftrightarrow \gamma(t)$ is zero for $t < 0$. The same holds for the time-domain admittance and impedance $\zeta(t) \leftrightarrow Z_{in}(s) = 1/Y_{in}(s)$. That $\gamma(t)$ and $\zeta(t)$ are causal is required by the physics.

The forward and retrograde waves are functions of frequency ω, as they depend on the source pressure (or velocity) and the point of horn excitation. The reflectance is a transfer function (thus the source term cancels) that depends only on the Thévenin impedance (or reflectance) looking into the system (at any position r).

To calculate $\Gamma(r, s)$ one must invert d'Alembert's superposition equation (Eq. 1.132)

$$\begin{bmatrix} \mathcal{P}^+(r,s) \\ \mathcal{P}^-(r,s) \end{bmatrix} = \frac{1}{2\mathcal{Y}(r)} \begin{bmatrix} \mathcal{Y}(r) & 1 \\ \mathcal{Y}(r) & -1 \end{bmatrix} \begin{bmatrix} \mathcal{P} \\ \mathcal{V} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & \mathcal{Z}(r) \\ 1 & -\mathcal{Z}(r) \end{bmatrix} \begin{bmatrix} \mathcal{P} \\ \mathcal{V} \end{bmatrix}. \tag{1.140}$$

The reflectance is defined as the ratio of the two pressure eigenfunctions

$$\Gamma(r,s) \equiv \frac{\mathcal{P}^-}{\mathcal{P}^+} = \frac{\mathcal{P} - \mathcal{Z}\,\mathcal{V}}{\mathcal{P} + \mathcal{Z}\,\mathcal{V}} = \frac{Z_{in} - \mathcal{Z}}{Z_{in} + \mathcal{Z}} = -\frac{Y_{in} - \mathcal{Y}}{Y_{in} + \mathcal{Y}}, \tag{1.141}$$

which is related to Eq. 1.135.

Given some experience with $Y_{in}(r, s)$ and $\Gamma(r, s)$, one soon appreciates the advantage of working with the reflectance over the radiation impedance/admittance $Z_{rad}(s)$ (aka immittance). The impedance has complicated properties, all of which are difficult to verify, whereas the reflectance is easily understood (it is closer to the physics). For example, we know that for a physical passive impedance $\Re Z \geq 0$. The corresponding property for the reflectance is $|\Gamma(\omega)| \leq 1$, with equality when the input resistance is zero.

It is important to note that because the area $A(x)$ is varying along the direction of propagation, energy is continuously being scattered back to the input, as captured by the area-dependent eigenfunctions. It is because of this scattering that the input admittance $Y_{in}(s)$ (Eq. 1.135) and the reflectance $\Gamma(r, s)$ (Eq. 1.141) depend on frequency, as explicitly shown in Fig. 1.21 (Morse, 1948, p. 283).